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Character Values of Groups of Odd Order and a Question of Feit

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Let χ be an irreducible complex character of a finite group G and let $Q(\chi)$ denote the field obtained by adjoining the values of χ to the rational field Q. Let Q_m denote the field obtained by adjoining a primitive mth root of unity to Q. We say that χ requires mth roots of unity if $Q(\chi)$ is contained in Q_m and m is the smallest positive integer with this property. The following open question is raised by Feit [2, p. 41]. Suppose that χ requires mth roots of unity. Is it true that G contains an element of order m? The purpose of this paper is to provide an affirmative answer to the question in the case that G has odd order.

In [1], Brauer showed that there is an affirmative answer to the question without restriction on the group order provided that each prime divisor of m occurs to the second power or more. We simply continue Brauer's argument to deal with the prime divisors that occur only to the first power. Using the fact that no nonidentity element can be conjugate to its inverse in a group of odd order, we are able to avoid difficulties that seem to obstruct a general proof. The final section contains a partial converse to the main theorem.

1. Preliminaries

Let p be a prime divisor of $|G|$ and let $|G| = n = hp^a$, where $(h, p) = 1$. We say that an irreducible character χ of G is p-rational if $Q(\chi)$ is contained in Q_h. Thus if χ requires mth roots of unity, it is not p-rational for any prime divisor p of m. The Galois group H of Q_n over Q_h is isomorphic to the Galois group of Q_{p^a} over Q and is thus cyclic if p is odd. It is well known that actions of H on the irreducible characters and classes of G can be defined and that these actions are permutation isomorphic when H is cyclic. This is the substance of Brauer's permutation lemma [2, p. 66]. The following easily proved lemma appears in [3].

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LEMMA 1. *Let G be a group of odd order $n = hp^a$, where $(h, p) = 1$. Let H be the Galois group of Q_n over Q_h. Let τ be the unique involution of H. An irreducible character of G is p-rational if and only if it is fixed by τ.*

We now derive a useful consequence of this lemma. Let $p_1, \ldots, p_r$ be distinct prime divisors of $|G|$, where $|G|$ is odd. Let h_i be the p_i'-part of G and let H_i be the Galois group of Q_n over Q_{h_i}. Let τ_i be the unique involution of H_i. The involution $\tau = \tau_1 \cdots \tau_r$ is an element of the Galois group of Q_n over Q. With this notation, we have the following result.

LEMMA 2. *An irreducible character χ of G is fixed by $\tau = \tau_1 \cdots \tau_r$ if and only if it is fixed by each of $\tau_1, \ldots, \tau_r$. Thus the irreducible characters fixed by τ are precisely the characters that are p_i-rational for all p_i.*

Proof. Let π denote the set of primes $p_1, \ldots, p_r$. We say that an irreducible character of G is π-rational if it is p-rational for each prime in π. An obvious generalization of the decomposition of an element into its p-part and its p'-part shows that any element g of G can be written uniquely in the form

$$g = xy = yx,$$

where the order of x is a π-number, and the order of y is a π'-number. It is easy to see that τ takes the class of g to the class of $x^{-1}y$. Since no nonidentity element of a group of odd order is conjugate to its inverse, τ fixes precisely the π-regular classes of G (that is, those classes whose elements have order coprime to each p_i). It follows from Brauer's permutation lemma that the number of irreducible characters fixed by τ equals the number of π-regular classes. Now τ clearly fixes the π-rational irreducible characters. However, Theorem 1.3 of [3] states that the number of π-regular classes equals the number of irreducible π-rational characters. Our lemma is thus proved.

2. STATEMENT AND PROOF OF MAIN THEOREM

THEOREM 1. *Let G be a group of odd order and let χ be an irreducible character of G that requires* mth *roots of unity. Then G contains an element of order m.*

Proof. Let $p_1, \ldots, p_r$ be the prime divisors of m that occur to the first power only and let $q_1, \ldots, q_s$ be the primes that divide m to the second power or more. Put $m = p_i k_i$, $1 \leqslant i \leqslant r$, and $m = q_i f_i$, $1 \leqslant i \leqslant s$. Let τ_i be the involution of the Galois group of Q_m over Q_{k_i} and let θ_i, of order q_i, generate the Galois group of Q_m over Q_{f_i}.

Following the proof of Corollary 3 of [1], we assert that a product σ of distinct τ_i and θ_j cannot fix χ. Since the θ_j have coprime order and do not fix χ, it is clear that σ cannot fix χ if it contains any θ_j. However, we saw in Lemma 2 that if a product of distinct τ_i fixes a character, each τ_i fixes the character. Since χ is not p_i-rational for any τ_i, it is not fixed by any τ_i. Thus σ cannot fix χ.

We now see that Case A of Brauer's Theorem 2, [1], applies. There must be an element g of G such that $\tau_i\chi(g) \neq \chi(g)$, $1 \leqslant i \leqslant r$, and $\theta_i\chi(g) \neq \chi(g)$, $1 \leqslant i \leqslant s$. The element g has order divisible by m, as required.

3. Partial Converse to Main Theorem

It is possible to gain a limited amount of information about the fields generated by the irreducible characters of a group G of odd order by examining the orders of the group elements. Let $p_1,..., p_r$ be distinct prime divisors of $|G|$ and let π denote the set of these primes. Let m be the π-part of $|G|$, and let k be the product of all the p_i. Let ε be a primitive mth root of unity and let σ be the automorphism that generates the Galois group of Q_m over Q_k, with $\sigma(\varepsilon) = \varepsilon^c$. With this notation, we have the following result.

Theorem 2. *The number of irreducible characters of G that require* $(p_1 \cdots p_r)$th *roots of unity equals the number of classes of π-elements x of G of order divisible by $p_1 \cdots p_r$ such that x is conjugate to x^c.*

Proof. We proceed by induction on r, the number of primes. Suppose that G has u classes of π-elements. It follows from Theorem 1.3 of [3] that G has exactly u irreducible characters $\chi_1,..., \chi_u$ whose values lie in Q_m. Moreover, by Theorem 1.4 of [3], the χ_i are linearly independent on the π-classes. Let H denote the Galois group of Q_m over Q_k. H permutes the classes of π-elements and the characters χ_i, and since H is cyclic, it follows from Brauer's permutation lemma that the number of χ_i fixed by H equals the number of π-classes fixed by H. We observe that the characters fixed by H are those whose values lie in Q_k, whereas a π-class containing x is fixed by H whenever x is conjugate to x^c.

Let π_1 be a proper subset of π and let q be the product of all the primes in π_1. We define $K(\pi_1)$ to be the set of all classes of π_1-elements x of order divisible by q such that x is conjugate to x^c. We note that $K(\pi_1)$ and $K(\pi_2)$ are disjoint if π_1, π_2 are distinct subsets of π. It is easy to see from the induction hypothesis that the number of irreducible characters that require qth roots of unity equals the number of classes in $K(\pi_1)$. Now the number of irreducible characters that require kth roots of unity equals the number of irreducible characters whose values lie in Q_k minus the number of irreducible characters which require qth roots of unity, q ranging over all possible products of less than r primes in π. Our evaluation of these numbers in the preceeding work yields the desired result for the number of characters that require kth roots of unity.

COROLLARY. *Let G be a group of odd order. Let $p_1, \ldots, p_r$ be distinct prime divisors of $|G|$ and let $k = p_1 \cdots p_r$. Suppose that G has u classes of elements of order k. Then G has at least u irreducible characters that require* kth *roots of unity.*

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